Interval Arithmetic for Graphic Processors

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Motivation

The pitfalls of floating point numbers

IEEE 754 floating point numbers

- Approximation of real numbers
- Rounding

$\Rightarrow$ Errors

- Cancellation
- Absorption
- Underflow
- ...

Intervals

Definition

The set

$$X = [\underline{x}\,,\,\overline{x}] := \{x \in \mathbb{R} | \underline{x} \leq x \leq \overline{x}\} \subseteq \mathbb{R}$$

with

- $\underline{x} \geq -\infty$
- $\overline{x} \leq +\infty$

is called an **interval**.

Definition

The set of all intervals including the empty set is denoted as $\overline{\mathbb{IR}}$.

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Operations on intervals

Definition

$$A \bullet B := \{a \circ b | a \in A, b \in B, \textit{if defined}\}$$

- $A, B \in \overline{\mathbb{IR}}$
- $\circ : \mathbb{R}^2 \to \mathbb{R}$ *continuous*

$$A + B := [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$$

$$A - B := [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$$

$$A \cdot B := [\min(\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b}), \max(\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b})]$$

$$A \div B := A \cdot [\frac{1}{\overline{b}}, \frac{1}{\underline{b}}] \qquad \text{with } 0 \notin B$$

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Fundamental theorem

Theorem (Inclusion property)

F, f are defined on $X \in \mathbb{IR}$

$$\Rightarrow f(X) \subseteq F(X)$$

- Function: $f : \mathbb{R} \to \mathbb{R}$
- Interval function: $F : \mathbb{IR} \to \mathbb{IR}$

Fields of application

- Root finding
- Solving systems of linear and non-linear equations
- Global optimization
- Computer graphics
- Computer assisted proofs
- Chemical engineering
- Electrical engineering
- Control theory
- Fluid mechanics
- ...

Root Finding

Interval Newton method

(Classical) Newton method:

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$$

- Non-convergence for some $x_0 \in [a\,,\,b]$

Interval Newton method:

$$N(X_n) := m(X_n) - \frac{f(m(X_n))}{F'(X_n)}$$

$$X_{n+1} := X_n \cap N(X_n)$$

- Idea: Use all tangents
- Reduce diameter
  - Use bisection otherwise
- Find all zeros

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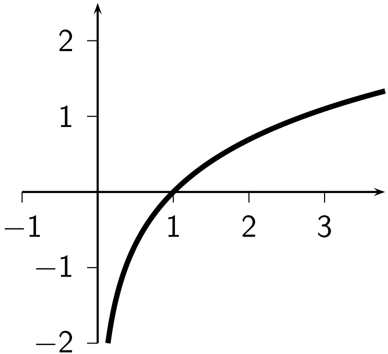
Interval Newton method

Illustration of the algorithm

1. Function f
2. Interval X_n
3. $0 \in F(X_n)$?
4. $N(X_n) := m(X_n) - \frac{f(m(X_n))}{F'(X_n)}$
5. $X_{n+1} := X_n \cap N(X_n)$



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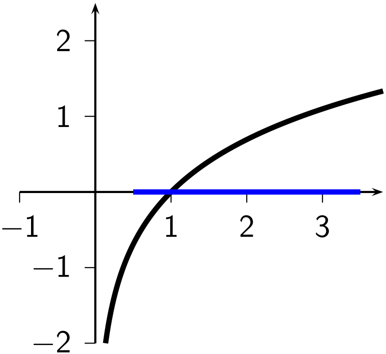

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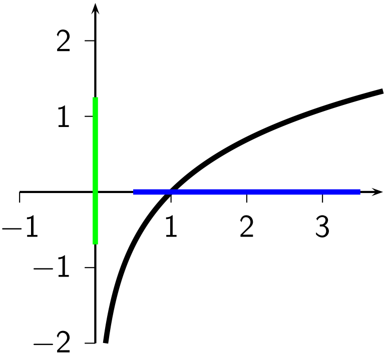

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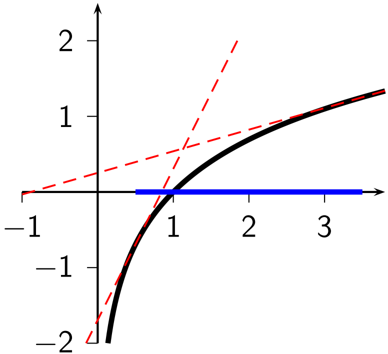

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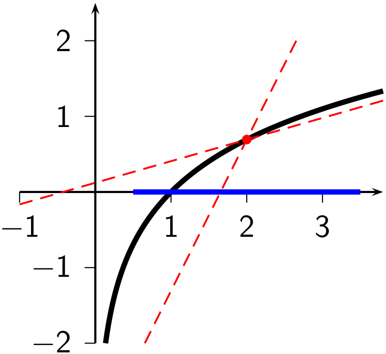

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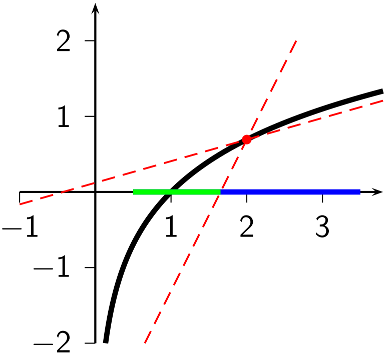
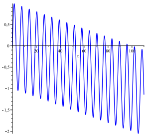


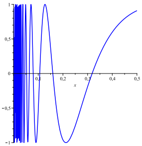
Our Work

Interval arithmetic for GPU's

Almost finished:

- Basic operations and elementary functions for CUDA
- Parallel interval Newton method for CUDA
  - Modification of the Algorithm
    - Load balancing
    - Partitioning
  - Depends on the function
    - Number of roots
    - Distribution





Our Work

Interval arithmetic for GPU's

Current:

- Solving systems of linear equations

Future:

- Global optimization

Questions ?

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